

# Hardy–type inequality in variable exponent Lebesgue spaces derived from nonlinear problem

Sylwia Dudek<sup>†</sup> and Iwona Skrzypczak<sup>\*◇</sup>

<sup>†</sup>Institute of Mathematics, Krakow University of Technology,  
ul. Warszawska 24, 31-155 Krakow, Poland  
e-mail: sbarnas@pk.edu.pl

<sup>◇</sup>Faculty of Mathematics, Informatics and Mechanics, University of Warsaw,  
ul. Banacha 2, 02-097 Warsaw, Poland  
e-mail: iskrzypczak@mimuw.edu.pl

## Abstract

We derive a family of weighted Hardy–type inequalities in the variable exponent Lebesgue space with an additional term of the form

$$\int_{\Omega} |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \mu_{3,\beta}(dx),$$

where  $\xi$  is any compactly supported Lipschitz function. The involved measures depend on a certain solution to the partial differential inequality involving  $p(x)$ –Laplacian  $-\Delta_{p(x)} u \geq \Phi$ , defined on an open and not necessarily bounded subset  $\Omega \subseteq \mathbb{R}^n$ , and a certain parameter  $\beta$ . We derive new Caccioppoli–type inequality for the solution  $u$ . As its consequence we get Hardy–type inequality.

We present the derivation of the family of weighted Hardy–type inequalities in  $\Omega \subseteq \mathbb{R}^n$ . We illustrate the result by several one-dimensional examples. The paper extends the recent results of the second author which imply classical Hardy and Hardy–Poincaré inequalities with the optimal constants.

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\*The author was supported by NCN grant 2011/03/N/ST1/00111.

**Keywords:**  $p(x)$ -Laplacian, Hardy inequality, Caccioppoli inequality, variable exponent Lebesgue spaces.

**2010 Mathematics Subject Classification:** 26D10, 35J60, 35J91.

## 1 Introduction

In this paper we derive a family of Hardy-type inequalities with variable exponent of the form

$$\int_{\Omega} |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \frac{|\nabla p(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (1)$$

where  $\Omega$  is an open subset of  $\mathbb{R}^n$ , not necessarily bounded, the exponent  $p$  is such that  $p \in W_{loc}^{1,1}(\Omega)$ ,  $p^{p(x)}, |\nabla p|^{p(x)} \in L_{loc}^1(\Omega)$  and satisfies  $1 < \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq \text{ess sup}_{x \in \Omega} p(x) < \infty$ , and a function  $\xi : \Omega \rightarrow \mathbb{R}$  is compactly supported and Lipschitz. The involved measures  $\mu_{1,\beta}(dx), \mu_{2,\beta}(dx)$  depend on  $p(x)$ , a certain parameter  $\beta$ , a continuous function  $\sigma(x)$ , and a nonnegative weak solution  $u$  to the PDI

$$-\Delta_{p(x)} u \geq \Phi \quad \text{in } \Omega, \quad (2)$$

with a locally integrable function  $\Phi$ . We admit the functions  $\sigma(x)$  and  $\Phi$  satisfying compatibility conditions with  $p(x)$  (see crucial conditions).

We deal with the variable exponent Lebesgue spaces, which recently have received more and more attention both — from the theoretical and from the applied point of view. We refer to [14, 30] for the detailed information on the theoretical approach to the Lebesgue and the Sobolev spaces with variable exponents. Various attempts to prove existence, uniqueness or regularity theory for problems stated in variable exponent spaces can be found e.g. in [3, 17]. We refer for the survey [21] summarising inter alia results on qualitative properties of solutions to the related PDEs. We mark that the variable exponent Lebesgue spaces are investigated since 1930s when Orlicz introduced them in [43]. They are under permanent development by various groups of mathematicians [10, 26, 41, 42].

The typical examples of equations stated in variable exponent spaces are models of electrorheological fluids, see e.g. [45, 46, 47]. This kind of materials have been intensively investigated recently. Electrorheological fluids change

their mechanical properties dramatically when an external electric field is applied, so the variable exponent Lebesgue setting is natural for their modelling. Some classical models are also generalised in the variable exponent Lebesgue spaces. In [14] we find investigations on Poisson equation, as well as Stokes problem being of fundamental importance in describing fluid dynamics. Let us mention that various models require different types of restrictions on  $p(x)$ , therefore the unified approach is missing.

Hardy-type inequalities are important tools in various fields of analysis. Let us mention such branches as functional analysis, harmonic analysis, probability theory, and PDEs. Hardy-type inequalities are investigated on their own in the classical way [31, 40, 49], as well as in the various generalised frameworks [4, 7, 8, 28, 51].

Recently, Hardy-type inequalities in the variable exponent Lebesgue spaces have become a lively studied topic of analysis [1, 11, 15, 20, 22, 23, 24, 25, 33, 34, 35, 36, 39, 37, 44, 48]. The most common idea in these papers is investigating links between validity of Hardy-type inequalities and boundedness of maximal operator. One-dimensional case is considered in [15, 36, 37], where the exponents are possibly different on the right- and the left-hand side of the inequality. The paper [48] is devoted to the inequality with the weights depending on distance from a single point, while in [20, 39] the weights depend on distance from a boundary in  $\mathbb{R}^n$ . There are several papers [11, 22, 23, 24, 25, 34, 35] dealing with the necessary and sufficient conditions for the validity of Hardy inequality involving Hardy operator. Different approach we find in [6], where the authors investigate the class of admissible weights for Hardy-type inequality holding for nonincreasing functions.

We point out that in the majority of the above papers the authors deal with the norm version of Hardy-type inequality. We obtain the modular one, which is stronger. We would like to stress that only in the constant exponent case the both types are equivalent. In the variable exponent case it is not direct to transform one of these types to another. To the authors' best knowledge the only result of this kind is given by Fan-Zhao [19, Theorem 1.3] where the authors derive a tool giving certain form of the norm version of Hardy inequality from a modular one.

The purpose of this paper is to introduce a new tool for derivation of Hardy-type inequalities with variable exponent on the basis of nonlinear problems. The idea of similar constructions in the constant exponent is present in a few papers. In [2] Barbatis, Filippas, and Tertikas derive Hardy-type inequalities on a domain where certain power of the function expressing

distance from the boundary is  $p$ -superharmonic. In [12] D'Ambrosio obtains an inequality related to (1) as a consequence of the inequality  $-\Delta_p(u^\alpha) \geq 0$  with a certain constant  $\alpha$ . Similar approach can be found in [49] by the second author.

Our considerations are based on the methods introduced in [29, 38] and developed in [49, 50, 51] in various ways. In [29] the authors investigate nonexistence of nontrivial nonnegative weak solutions to  $A$ -harmonic problems starting with derivation of Caccioppoli-type estimate for their weak solutions. As a starting point to derive Hardy-type inequality we focus on this step. We modify the proof of Theorem 4.1 from [49], where the investigated PDI reads

$$-\Delta_p u \geq \Phi \quad \text{in } \Omega, \quad (3)$$

with a locally integrable function  $\Phi$  being in a certain sense not very negative. This condition generalises the requirement that the solution  $u$  is supposed to be a  $p$ -superharmonic function. As it is shown in [49], the substitution in the derived Caccioppoli-type inequality for solutions implies the family of Hardy-type inequalities of the form

$$\int_{\Omega} |\xi|^p \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_{2,\beta}(dx),$$

where  $1 < p < \infty$ ,  $\xi : \Omega \rightarrow \mathbb{R}$  is compactly supported Lipschitz function, and  $\Omega$  is an open subset of  $\mathbb{R}^n$ . The involved measures  $\mu_{1,\beta}(dx)$ ,  $\mu_{2,\beta}(dx)$  depend on a certain parameter  $\beta$  and on  $u$  — a nonnegative weak solution to (3). Among other results it implies classical Hardy and Hardy–Poincaré inequalities with optimal constants (see [49, 50], respectively). We retrieve the main result of [49] as a special case here (see Theorem 6.1) and therefore we confirm all the examples from [49, 50].

We extend the techniques from [49] to the more general case when we deal with (2) instead of (3). We derive Hardy-type inequality in the variable exponent Lebesgue spaces on  $\mathbb{R}^n$ . Then we pay particular attention to the case of  $n = 1$ , because it is easier to compare with many existing one-dimensional results, e.g. [6, 15, 20, 36, 37]. Moreover, higher dimensional problems may be reduced to this case, when we assume certain kind of symmetry. The paper [16] is devoted to further analysis of the results of our paper in  $\mathbb{R}^n$ . We hope that our result will be found useful in applied mathematics, especially in investigations on qualitative properties of solutions to nonlinear problems.

The paper is organised as follows. Section 3 is devoted to derivation of Caccioppoli-type inequality for solutions to (2). In Section 4 we derive

general  $p(x)$ -Hardy inequality for compactly supported Lipschitz functions. In Section 5 we concentrate on inequalities in one dimension. In Section 6 we give detailed comparison with the results existing in the literature. We conclude our paper in Section 7 by posing open questions.

## 2 Preliminaries

### Notation

In the sequel we assume that  $\Omega \subseteq \mathbb{R}^n$  is an open subset not necessarily bounded. If  $f$  is defined on the set  $A$  by  $f\chi_A$  we understand function  $f$  extended by 0 outside  $A$ . By  $\langle \cdot, \cdot \rangle$  we understand the classical scalar product in  $\mathbb{R}^n$ . We say that the function  $f$  has values separated from 0, if there exists a constant  $c_0$  such that  $f(x) \geq c_0 > 0$  for every  $x$ .

### General Lebesgue and Sobolev spaces

In the sequel we suppose that measurable function  $p : \Omega \rightarrow (1, \infty)$  is such that

$$1 < p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x) \leq p(x) \leq p^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x) < \infty. \quad (4)$$

We recall some properties of the variable exponent spaces  $L^{p(x)}(\Omega)$  and  $W^{1,p(x)}(\Omega)$ . By  $E(\Omega)$  we denote the set of all equivalence classes of measurable real functions defined on  $\Omega$  being equal almost everywhere. The variable exponent Lebesgue space is defined as

$$L^{p(x)}(\Omega) = \{u \in E(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < \infty\}$$

equipped with the Luxemburg-type norm

$$\|u\|_{L^{p(x)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega; \mathbb{R}^n)\}$$

equipped with the norm  $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}$ .

Then  $(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$  and  $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$  are separable and reflexive Banach spaces.

For more detailed information we refer to [14, 18, 19].

By  $\mathcal{P}(\Omega)$  we denote the class of the functions  $p$  such that (4) is satisfied and  $p \in W_{loc}^{1,1}(\Omega)$ ,  $p^{p(x)}, |\nabla p|^{p(x)} \in L_{loc}^1(\Omega)$ .

### Differential inequality

Our analysis is based on the following differential inequality.

**Definition 2.1.** *Let  $\Omega$  be any open subset of  $\mathbb{R}^n$ . We assume that the measurable function  $p : \Omega \rightarrow (1, \infty)$  satisfies (4) and  $\Phi$  is the locally integrable function defined in  $\Omega$  such that for every nonnegative compactly supported  $w \in W^{1,p(x)}(\Omega)$ , we have  $\int_{\Omega} \Phi w \, dx > -\infty$ .*

*Let  $u \in W_{loc}^{1,p(x)}(\Omega)$  and  $u \not\equiv 0$ . We say that*

$$-\Delta_{p(x)} u \geq \Phi,$$

*if for every nonnegative compactly supported  $w \in W^{1,p(x)}(\Omega)$ , we have*

$$\langle -\Delta_{p(x)} u, w \rangle := \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla w \rangle \, dx \geq \int_{\Omega} \Phi w \, dx. \quad (5)$$

**Remark 2.1.** Note that  $p(x)$ -Laplacian is a continuous, bounded, and strictly monotone operator defined for every compactly supported function  $w \in W^{1,p(x)}(\Omega)$  (see e.g. [17, Theorem 3.1] for the definitions and the proofs). In particular, it is well-defined in the distributional sense.

### Crucial conditions

We suppose that the measurable function  $p : \Omega \rightarrow (1, \infty)$  satisfies (4), non-negative  $u \in W_{loc}^{1,p(x)}(\Omega)$  and  $\Phi \in L_{loc}^1(\Omega)$  satisfy PDI  $-\Delta_{p(x)} u \geq \Phi$ , in the sense of Definition 2.1. We assume that there exist a continuous function  $\sigma(x) : \overline{\Omega} \rightarrow \mathbb{R}$  and a parameter  $\beta > 0$ , such that the following conditions are satisfied

$$\Phi \cdot u + \sigma(x) |\nabla u|^{p(x)} \geq 0 \quad \text{a.e. in } \Omega, \quad (6)$$

$$\beta > \sup_{x \in \overline{\Omega}} \sigma(x). \quad (7)$$

### 3 Caccioppoli estimate for solution of differential inequality $-\Delta_{p(x)}u \geq \Phi$

Before we formulate the main theorem of this section we state the following useful lemmas.

**Lemma 3.1.** *Let  $u \in W_{loc}^{1,p(x)}(\Omega)$ ,  $u > 0$  and  $\phi$  be a nonnegative Lipschitz function with compact support in  $\Omega$  such that the integral  $\int_{\text{supp } \phi} |\nabla \phi|^{p(x)} \phi^{1-p(x)} dx$  is finite. We fix  $0 < \delta < R$ ,  $\beta > 0$  and denote*

$$u_{\delta,R}(x) := \min \{u(x) + \delta, R\}, \quad G(x) := (u_{\delta,R}(x))^{-\beta} \phi(x). \quad (8)$$

Then  $u_{\delta,R} \in W_{loc}^{1,p(x)}(\mathbb{R}^n)$  and  $G \in W^{1,p(x)}(\Omega)$ .

**Remark 3.1.** See e.g. [14, Proposition 8.1.9], to obtain  $u_{\delta,R} \in W_{loc}^{1,p(x)}(\mathbb{R}^n)$ . We note that the truncated function satisfies  $\delta \leq u_{\delta,R}(x) \leq R$  and therefore we have  $(u_{\delta,R}(x))^{-\beta} \in W_{loc}^{1,p(x)}(\mathbb{R}^n)$ . The function  $G$  is compactly supported, thus  $G \in W^{1,p(x)}(\Omega)$ .

**Lemma 3.2.** *Let a measurable function  $p : \Omega \rightarrow (1, \infty)$  satisfy (4), a function  $\tau(x) : \Omega \rightarrow \mathbb{R}_+$  be continuous, bounded, with values separated from 0, and  $s_1, s_2 \geq 0$ . Then for a.e.  $x \in \Omega$  we have*

$$s_1 s_2^{p(x)-1} \leq \frac{1}{p(x)\tau(x)^{p(x)-1}} \cdot s_1^{p(x)} + \frac{p(x)-1}{p(x)} \tau(x) \cdot s_2^{p(x)}.$$

**Proof.** We apply classical Young inequality  $ab \leq \frac{a^{p(x)}}{p(x)} + \frac{p(x)-1}{p(x)} b^{\frac{p(x)}{p(x)-1}}$  with  $a = \frac{s_1}{\eta(x)^{p(x)-1}}$ ,  $b = (s_2 \eta(x))^{p(x)-1}$ , where  $\eta(x)$  is an arbitrary continuous, bounded function with values separated from 0, to get

$$\begin{aligned} s_1 s_2^{p(x)-1} &= \left( \frac{s_1}{\eta^{p(x)-1}} \right) (s_2 \eta)^{p(x)-1} \leq \\ &\leq \frac{1}{p(x)} \left( \frac{s_1}{\eta^{p(x)-1}} \right)^{p(x)} + \frac{p(x)-1}{p(x)} (s_2 \eta)^{(p(x)-1) \frac{p(x)}{p(x)-1}} = \\ &= \frac{1}{p(x) \eta^{p(x)(p(x)-1)}} \cdot s_1^{p(x)} + \frac{p(x)-1}{p(x)} \eta^{p(x)} \cdot s_2^{p(x)}. \end{aligned}$$

Now it suffices to substitute  $\tau(x) = \eta(x)^{p(x)}$ . □

**Lemma 3.3.** *Let  $u \in W_{loc}^{1,1}(\Omega)$  be defined everywhere by the formula (see e.g. [5])*

$$u(x) := \limsup_{r \rightarrow 0} \int_{B(x,r)} u(y) dy$$

*and let  $t \in \mathbb{R}$ . Then*

$$\{x \in \mathbb{R}^n : u(x) = t\} \subseteq \{x \in \mathbb{R}^n : \nabla u(x) = 0\} \cup N,$$

*where  $N$  is a set of Lebesgue's measure zero.*

The main goal of this section is the following result.

**Theorem 3.1** (Caccioppoli estimate). *Let  $\Omega \subseteq \mathbb{R}^n$  be an open subset. We suppose that the measurable function  $p : \Omega \rightarrow (1, \infty)$  satisfy (4), nonnegative  $u \in W_{loc}^{1,p(x)}(\Omega)$  and  $\Phi \in L_{loc}^1(\Omega)$  satisfy  $PDI - \Delta_{p(x)} u \geq \Phi$ , in the sense of Definition 2.1. Assume further that functions  $u$ ,  $\Phi$ ,  $p(x)$ ,  $\sigma(x)$  and a parameter  $\beta > 0$  satisfy crucial conditions (6) and (7).*

*Then the inequality*

$$\begin{aligned} & \int_{\Omega} (\Phi \cdot u + \sigma(x) |\nabla u|^{p(x)}) u^{-\beta-1} \chi_{\{u>0\}} \cdot \phi \, dx \leq \\ & \leq \int_{\Omega} \frac{(p(x) - 1)^{p(x)-1}}{(p(x))^{p(x)} (\beta - \sigma(x))^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx, \end{aligned} \quad (9)$$

*holds for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$  such that the integral  $\int_{\text{supp } \phi} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx$  is finite.*

We call (9) Caccioppoli estimate, because it involves  $\nabla u$  on the left-hand side and, when we estimate  $\chi_{\{\nabla u \neq 0\}} \leq 1$  on the right-hand side, then the right-hand side depends only on  $u$  (see e.g. [9, 27]).

We note that we do not assume that the right-hand side in (9) is finite.

The proof is based on the idea of the proof of Theorem 3.1 from [49] whose further inspiration is the proof of Proposition 3.1 from [29].

**Proof of Theorem 3.1.** The proof follows by three steps.



**Step 1. Derivation of a local inequality.**

We obtain the following lemma.

**Lemma 3.4.** *We suppose that the measurable function  $p : \Omega \rightarrow (1, \infty)$  satisfy (4), nonnegative  $u \in W_{loc}^{1,p(x)}(\Omega)$  and  $\Phi \in L_{loc}^1(\Omega)$  satisfy PDI  $-\Delta_{p(x)}u \geq \Phi$ , in the sense of Definition 2.1. Assume further that  $\beta > 0$  is arbitrary number and  $\varepsilon(x)$  is a bounded function with values separated from 0.*

*Then, for every  $0 < \delta < R$ , the inequality*

$$\begin{aligned} & \int_{\Omega} \left( \Phi \cdot (u + \delta) + \left( \beta - \frac{p(x)-1}{p(x)} \varepsilon(x) \right) |\nabla u|^{p(x)} \right) (u + \delta)^{-\beta-1} \chi_{\{u \leq R-\delta\}} \cdot \phi \, dx \quad (10) \\ & \leq \int_{\Omega} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u \leq R-\delta\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + C(\delta, R), \end{aligned}$$

where

$$C(\delta, R) = R^{-\beta} \left[ \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \chi_{\{\nabla u \neq 0, u > R-\delta\}} \, dx - \int_{\Omega} \Phi \chi_{\{u > R-\delta\}} \phi \, dx \right] \quad (11)$$

holds for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$ .

**Proof of Lemma 3.4.** We take  $w = G$  (see (8)) in the left side of the inequality (5) and note that

$$\begin{aligned} L &:= \int_{\Omega} \Phi \cdot G \, dx = \int_{\Omega} \Phi \cdot (u_{\delta,R})^{-\beta} \phi \, dx = \quad (12) \\ &= \int_{\Omega \cap \{u \leq R-\delta\}} \Phi \cdot (u + \delta)^{-\beta} \phi \, dx + R^{-\beta} \int_{\Omega \cap \{u > R-\delta\}} \Phi \cdot \phi \, dx. \end{aligned}$$

On the other hand, inequality (5) implies

$$\begin{aligned} L &:= \int_{\Omega} \Phi \cdot G \, dx \leq \langle -\Delta_{p(x)}u, G \rangle = \int_{\Omega \cap \{\nabla u \neq 0\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla G \rangle \, dx = \\ &= -\beta \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi \, dx + \\ &+ \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} \, dx + \\ &+ R^{-\beta} \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \, dx. \end{aligned}$$

Note that all the above integrals are finite, what follows from Lemma 3.1 (for  $0 \leq u \leq R - \delta$  we have  $\delta \leq u + \delta \leq R$ ). We compute further that

$$\begin{aligned}
 & \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} dx \leq \\
 & \leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} |\nabla u|^{p(x)-1} |\nabla \phi| (u + \delta)^{-\beta} dx = \\
 & = \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \left( \frac{|\nabla \phi|}{\phi} (u + \delta) \right) \cdot |\nabla u|^{p(x)-1} (u + \delta)^{-\beta-1} \phi dx.
 \end{aligned}$$

We apply Lemma 3.2 with  $s_1 = \frac{|\nabla \phi|}{\phi} (u + \delta)$ ,  $s_2 = |\nabla u|$  and an arbitrary bounded and continuous function  $\tau(x) = \varepsilon(x) > 0$  with values separated from 0, to get

$$\begin{aligned}
 & \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle (u + \delta)^{-\beta} dx \leq \\
 & \leq \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{p(x) - 1}{p(x)} \varepsilon(x) |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi dx + \\
 & + \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{1}{p(x) \varepsilon(x)^{p(x)-1}} \left( \frac{|\nabla \phi|}{\phi} \right)^{p(x)} (u + \delta)^{p(x)-\beta-1} \phi dx.
 \end{aligned}$$

Combining these estimates we deduce that

$$\begin{aligned}
 L & \leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} \left( -\beta + \frac{p(x) - 1}{p(x)} \varepsilon(x) \right) |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi dx + \\
 & + \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{1}{p(x) \varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} |\nabla \phi|^{p(x)} \phi^{1-p(x)} dx + \\
 & + R^{-\beta} \int_{\Omega \cap \{\nabla u \neq 0, u > R - \delta\}} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle dx.
 \end{aligned}$$

This and (12) imply

$$\begin{aligned}
 & \int_{\Omega \cap \{u \leq R - \delta\}} \Phi \cdot (u + \delta)^{-\beta} \phi dx + \\
 & + \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} \left( \beta - \frac{p(x) - 1}{p(x)} \varepsilon(x) \right) |\nabla u|^{p(x)} (u + \delta)^{-\beta-1} \phi dx \leq \\
 & \leq \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R - \delta\}} \frac{1}{p(x) \varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} |\nabla \phi|^{p(x)} \phi^{1-p(x)} dx + C(\delta, R),
 \end{aligned}$$

where  $C(\delta, R)$  is given by (11).  $\square$

**Remark 3.1.** Introduction of parameters  $\delta$  and  $R$  was necessary as we needed to move some finite quantities in the estimates to opposite sides of inequalities.

**Step 2. Passing to the limit with  $\delta \searrow 0$ .**

We show that when  $\beta > 0$  is an arbitrary number,  $\varepsilon(x)$  is a bounded function with values separated from 0, such that  $\beta - \frac{p(x)-1}{p(x)}\varepsilon(x) =: \sigma(x)$ , then for any  $R > 0$

$$\begin{aligned} & \int_{\Omega} (\Phi \cdot u + \sigma(x) |\nabla u|^{p(x)}) u^{-\beta-1} \chi_{\{0 < u \leq R\}} \cdot \phi \, dx \\ & \leq \int_{\Omega} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx + C(R), \end{aligned} \quad (13)$$

where

$$C(R) = R^{-\beta} \left[ \int_{\Omega} |\nabla u|^{p(x)-2} |\nabla u| \chi_{\{u \geq \frac{R}{2}\}} \cdot |\nabla \phi| \, dx + \int_{\Omega} \Phi \chi_{\{u \geq \frac{R}{2}\}} \cdot \phi \, dx \right]$$

holds for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$  such that the integral  $\int_{\text{supp } \phi \cap \{\nabla u \neq 0\}} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx$  is finite. Moreover, all quantities appearing in (13) are finite.

We show first that under our assumptions, when  $\delta \searrow 0$ , we have

$$\begin{aligned} & \int_{\Omega} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx \rightarrow \\ & \rightarrow \int_{\Omega} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{\nabla u \neq 0, u \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx \end{aligned} \quad (14)$$

for every nonnegative Lipschitz function  $\phi$  with compact support in  $\Omega$  such that the integral  $\int_{\text{supp } \phi \cap \{\nabla u \neq 0\}} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx$  is finite.

We note that  $(u + \delta)^{p(x)-\beta-1} \chi_{\{u + \delta \leq R\}} \xrightarrow{\delta \rightarrow 0} u^{p(x)-\beta-1} \chi_{\{u \leq R\}}$  a.e. This follows from Lemma 3.3 (which gives that the set  $\{u = 0, |\nabla u| \neq 0\}$  is of measure zero) and the continuity outside zero of the involved functions.

We show (14) independently on separate subsets of domains of integration. Hence, we have

$$\begin{aligned} & \int_{\Omega \cap \{\nabla u \neq 0\}} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} \chi_{\{u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx = \\ & = \sum_{i=1}^3 \int_{E_i \cap \{\nabla u \neq 0\}} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} \chi_{\{u + \delta \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} \, dx, \end{aligned}$$

where

$$\begin{aligned} E_1 &= \{x \in \Omega : p(x) - \beta - 1 = 0\}, \\ E_2 &= \{x \in \Omega : p(x) - \beta - 1 < 0\}, \\ E_3 &= \{x \in \Omega : p(x) - \beta - 1 > 0\}. \end{aligned}$$

Convergence on  $E_1$  follows from the Lebesgue's Monotone Convergence Theorem, as on this set the only expression involving  $\delta$  is the characteristic function  $\chi_{\{u+\delta \leq R\}}$ .

Let us concentrate on the case when  $\delta \searrow 0$  on  $E_2$ . We apply the Lebesgue's Monotone Convergence Theorem as on this set

$$(u + \delta)^{p(x)-\beta-1} \chi_{\{u+\delta \leq R\}} \nearrow u^{p(x)-\beta-1} \chi_{\{u \leq R\}}.$$

Indeed, we note first that then for a.e.  $x \in \Omega$  such that  $u(x) > 0$  we have that  $u + \delta \searrow u$ . Hence, also  $(u + \delta)^{p(x)-\beta-1} \nearrow u^{p(x)-\beta-1} \neq 0$ . Secondly, we observe that then for a.e.  $x \in \Omega$  we have  $\chi_{\{0 \leq u \leq R-\delta\}} \leq \chi_{\{0 < u \leq R-\delta\}} \nearrow \chi_{\{0 < u < R\}}$ .

In the case of  $E_3$ , without loss of generality, we assume that  $R > 1$ . Then we apply the Lebesgue's Dominated Convergence Theorem as

$$\begin{aligned} \int_{E_3 \cap \{\nabla u \neq 0\}} \frac{1}{p(x)\varepsilon(x)^{p(x)-1}} (u + \delta)^{p(x)-\beta-1} \chi_{\{u+\delta \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} dx &\leq \\ &\leq R^{p^+-\beta-1} \frac{\tilde{\varepsilon}}{p^-} \int_{E_3 \cap \{\nabla u \neq 0\}} \chi_{\{u \leq R\}} \cdot |\nabla \phi|^{p(x)} \phi^{1-p(x)} dx < \infty, \end{aligned}$$

where  $\tilde{\varepsilon} = \sup_{x \in \Omega} [\varepsilon(x)^{1-p(x)}]$ . The details are left to the reader.

To complete the proof of Step 2 we note that (14) says that, when  $\delta \searrow 0$ , the first integral on the right-hand side of (10) is convergent to the first integral of the right-hand side of (13). To deal with the second expression note that for  $\delta \leq \frac{R}{2}$ , we have

$$\begin{aligned} |C(\delta, R)| &\leq \left| R^{-\beta} \int_{\Omega} |\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle \chi_{\{u > R-\delta\}} dx \right| + \\ &\quad + \left| R^{-\beta} \int_{\Omega} \Phi \chi_{\{u > R-\delta\}} \cdot \phi dx \right| \leq C(R). \end{aligned}$$

It suffices now to pass to the limit with  $\delta \searrow 0$  on the left-hand side of (10). We do it due to the Lebesgue's Monotone Convergence Theorem as the expression in brackets is nonnegative and decreasing. Indeed, the condition (6) implies

$$\Phi \cdot (u + \delta) + \sigma(x) |\nabla u|^{p(x)} \geq \Phi \cdot u + \sigma(x) |\nabla u|^{p(x)} \geq 0 \quad \text{a.e. on } \Omega \cap \{u > 0\}.$$

**Step 3.** We let  $R \rightarrow \infty$  and finish the proof.

Without loss of generality we can assume that the integral in the right-hand side of (9) is finite, as otherwise the inequality follows trivially. Note that since  $|\nabla u|^{p(x)-2} \langle \nabla u, \nabla \phi \rangle$  and  $\Phi \phi$  are integrable we have  $\lim_{R \rightarrow \infty} C(R) = 0$ . Therefore, (9) follows from (13) by the Lebesgue's Monotone Convergence Theorem (note that  $\varepsilon(x) = \frac{p(x)(\beta - \sigma(x))}{p(x)-1}$  by the choice of  $\sigma(x)$ ).  $\square$

## 4 General $p(x)$ -Hardy inequality

In the proof of  $p(x)$ -Hardy inequality we need the following lemma.

**Lemma 4.1.** *Let  $p : \Omega \rightarrow (1, \infty)$  satisfy (4) and  $s_1, s_2 \geq 0$ , then the following inequality holds for a.e.  $x \in \Omega$*

$$(s_1 + s_2)^{p(x)} \leq 2^{(p(x)-1)\chi_{\{s_1 \neq 0\}}} \left( s_1^{p(x)} + s_2^{p(x)} \right). \quad (15)$$

**Remark 4.1.** Note that in this lemma the role of  $s_1$  is not the same as  $s_2$ . If  $s_1 = 0$ , then (15) becomes  $s_2^{p(x)} = s_2^{p(x)}$ . This is necessary to retrieve Theorem 4.1 from [49] (concerning constant exponent case) with the best constant via our investigations (see Theorem 6.1 here).

Now we state our main result.

**Theorem 4.1** ( $p(x)$ -Hardy inequality). *Let  $\Omega \subseteq \mathbb{R}^n$  be an open subset not necessarily bounded and  $p \in \mathcal{P}(\Omega)$ . Let nonnegative  $u \in W_{loc}^{1,p(x)}(\Omega)$  and  $\Phi \in L_{loc}^1(\Omega)$  satisfy  $PDI - \Delta_{p(x)} u \geq \Phi$ , in the sense of Definition 2.1. Assume further that functions  $u$ ,  $\Phi$ ,  $p(x)$ ,  $\sigma(x)$  and a parameter  $\beta > 0$  satisfy crucial conditions (6) and (7).*

*Then for every Lipschitz function  $\xi$  with compact support in  $\Omega$  we have*

$$\int_{\Omega} |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \cdot \frac{|\nabla p(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (16)$$

where

$$\mu_{1,\beta}(dx) = (\Phi \cdot u + \sigma(x) |\nabla u|^{p(x)}) \cdot u^{-\beta-1} \chi_{\{u > 0\}} dx, \quad (17)$$

$$\mu_{2,\beta}(dx) = \left( \frac{p(x)-1}{\beta - \sigma(x)} \right)^{p(x)-1} 2^{(p(x)-1)\chi_{\{|\nabla p| \neq 0\}}} u^{p(x)-\beta-1} \chi_{\{|\nabla u| \neq 0\}} dx. \quad (18)$$

*Proof.* We are going to apply Theorem 3.1 and, after substituting a certain form of function  $\phi$ , we estimate the right-hand side of (9).

We take  $\xi(x) = (\phi(x))^{\frac{1}{p(x)}}$ . Then whenever  $\phi > 0$ , we have

$$\nabla \xi = \frac{1}{p(x)} \phi^{\frac{1}{p(x)}-1} \nabla \phi - \frac{\log \phi}{p^2(x)} \phi^{\frac{1}{p(x)}} \nabla p(x).$$

Equivalently, we have

$$\phi^{\frac{1}{p(x)}-1} \nabla \phi = p(x) \nabla \xi + \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x). \quad (19)$$

We observe that

$$\left\{ \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} |\nabla p(x)| \neq 0 \right\} \subseteq \{ |\nabla p(x)| \neq 0 \} =: P.$$

We apply Lemma 4.1 to (19) (with  $s_1 = \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} |\nabla p(x)|$  and  $s_2 = p(x) \nabla \xi$ ) to get

$$\begin{aligned} \left| \phi^{\frac{1}{p(x)}-1} \nabla \phi \right|^{p(x)} &= \left| p(x) \nabla \xi + \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x) \right|^{p(x)} \leq \\ &\leq 2^{(p(x)-1)\chi_P} |p(x) \nabla \xi|^{p(x)} + 2^{(p(x)-1)\chi_P} \left| \frac{\log \phi}{p(x)} \phi^{\frac{1}{p(x)}} \nabla p(x) \right|^{p(x)}. \end{aligned} \quad (20)$$

We substitute  $\xi^{p(x)} = \phi$  on the right-hand side of (20) to obtain

$$\begin{aligned} |\nabla \phi|^{p(x)} \phi^{1-p(x)} &= \left| \phi^{\frac{1}{p(x)}-1} \nabla \phi \right|^{p(x)} \leq \\ &\leq 2^{(p(x)-1)\chi_P} |p(x) \nabla \xi|^{p(x)} + 2^{(p(x)-1)\chi_P} \left| \frac{\log(\xi^{p(x)})}{p(x)} \xi \nabla p(x) \right|^{p(x)} = \\ &= 2^{(p(x)-1)\chi_P} |p(x) \nabla \xi|^{p(x)} + 2^{(p(x)-1)\chi_P} |\xi \log \xi \nabla p(x)|^{p(x)}. \end{aligned} \quad (21)$$

We recall that  $\mu_{1,\beta}$  is given in (17) and let us denote  $\mu$  as follows

$$\mu(dx) = \frac{(p(x)-1)^{p(x)-1}}{p(x)^{p(x)}(\beta - \sigma(x))^{p(x)-1}} u^{p(x)-\beta-1} \chi_{\{|\nabla u| \neq 0\}} dx.$$

Applying (21), we get

$$\begin{aligned} \int_{\Omega} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \mu(dx) &= \int_{\Omega} \left| \phi^{\frac{1}{p(x)}-1} \nabla \phi \right|^{p(x)} \mu(dx) \leq \\ &\leq \int_{\Omega} 2^{(p(x)-1)\chi_P} \left( |p(x) \nabla \xi|^{p(x)} + |\xi \log \xi \nabla p(x)|^{p(x)} \right) \mu(dx) = \\ &= \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \frac{|\nabla p(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \end{aligned}$$

where  $\mu_{2,\beta}(dx)$  is given by (18).

Summing up, by Theorem 3.1, we obtain

$$\begin{aligned} \int_{\Omega} \xi^{p(x)} \mu_{1,\beta}(dx) &= \int_{\Omega} \phi \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \phi|^{p(x)} \phi^{1-p(x)} \mu(dx) \leq \\ &\leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \frac{|\nabla p(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \end{aligned}$$

which completes the proof.  $\square$

## 5 One-dimensional case

This section is devoted to the case when  $\Omega = I \subseteq \mathbb{R}$  is an open interval (not necessarily finite). We give here a few original examples indicating that our conditions on admissible functions  $p(x)$  are not very restrictive.

Let us start with the following direct corollary of Theorem 4.1.

**Corollary 5.1.** *Suppose  $I \subseteq (-M, M) \subseteq \mathbb{R}$ , with some  $M > 0$ , is a bounded open subset,  $u = M - |x|$ , and  $p \in \mathcal{P}(I)$ . Assume further that nonnegative  $\sigma(x)$  and  $\beta > 0$  satisfy crucial condition (7).*

*Then for every Lipschitz function  $\xi$  with compact support in  $I$ , we have*

$$\int_I |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |\xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx),$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= (M - |x|)^{-\beta-1} \sigma(x) dx, \\ \mu_{2,\beta}(dx) &= (M - |x|)^{p(x)-\beta-1} \left[ 2 \cdot \frac{p(x) - 1}{\beta - \sigma(x)} \right]^{p(x)-1} dx. \end{aligned}$$

*Proof.* We apply  $u = M - |x|$  on  $\Omega = I = (-M, M)$  in Theorem 4.1. In this case  $u' = -\text{sgn}(x)$ ,  $u'' \equiv 0$  outside 0 and  $u''(0) = -2\delta_0$  is  $-2$  times Dirac delta. It enables to choose  $\Phi \equiv 0$  in the PDI  $-\Delta_{p(x)} u \geq \Phi$  due to Definition 2.1. Direct computations finish the proof.  $\square$

When the considered solution to nonlinear problem is more regular, one-dimensional version of Theorem 4.1 can be reduced in the following way.

**Theorem 5.1** (One-dimensional inequality). *Let  $I \subseteq \mathbb{R}$ ,  $p \in \mathcal{P}(I)$ , and  $u \in W_{loc}^{1,p(x)}(I) \cap W_{loc}^{2,1}(I)$  be a nonnegative function, such that  $|u|^{p(x)-2}u' \in W_{loc}^{1,1}(I)$ . Assume further that  $\sigma(x)$  and  $\beta > 0$  satisfy crucial condition (7) and the following condition is satisfied*

$$g(x) := \sigma(x)(u')^2 - p'(x)uu' \log |u'| - (p(x) - 1)uu'' \geq 0 \quad \text{a.e. } x \in I. \quad (22)$$

*Then, for every Lipschitz function  $\xi$  with compact support in  $I$ , we have*

$$\int_I |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |\xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (23)$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= \frac{|u'|^{p(x)-2}}{u^{\beta+1}} g(x) \chi_{\{u>0\}} dx, \\ \mu_{2,\beta}(dx) &= \left( \frac{p(x) - 1}{\beta - \sigma(x)} \right)^{p(x)-1} 2^{(p(x)-1)\chi_{\{p' \neq 0\}}} u^{p(x)-\beta-1} \chi_{\{|u'| \neq 0\}} dx. \end{aligned}$$

*Proof.* It suffices to apply Theorem 4.1 with  $u = u(x)$ ,  $x \in I$ . Suppose  $\tilde{I}$  is the set where  $u''$  is well defined, then

$$\Delta_{p(x)} u = (|u'|^{p(x)-2}u')' = (|u'|^{p(x)-2})'u' + |u'|^{p(x)-2}u'' \quad \text{on } \tilde{I}$$

and thus

$$-\Delta_{p(x)} u = -|u'|^{p(x)-2} [p'(x) \cdot u' \log |u'| + (p(x) - 2)u'' + u''] \quad \text{on } \tilde{I}.$$

We set

$$\Phi = \begin{cases} -\Delta_{p(x)} u & \text{if } u \in \tilde{I}, \\ 0 & \text{if } u \in I \setminus \tilde{I}, \end{cases}$$

which satisfies all the restrictions of Theorem 4.1. Direct computations gives inequality (23).  $\square$

We illustrate the above theorem by several examples. We give below the inequality with power-type weights, where we allow  $I = (0, \infty)$ .

**Corollary 5.2.** *Suppose  $I \subseteq \mathbb{R}_+$  is an open subset,  $p \in \mathcal{P}(I)$ , and  $\alpha \in \mathbb{R}$  is an arbitrary number. Assume further that  $\sigma(x)$  and  $\beta > 0$  satisfy crucial condition (7) and the following condition is satisfied*

$$\bar{g}(x) := \sigma(x)\alpha^2 - p'(x)x\alpha \log |\alpha x^{\alpha-1}| + (p(x) - 1)\alpha(1 - \alpha) \geq 0 \quad \text{a.e. } x \in I. \quad (24)$$



Then, for every Lipschitz function  $\xi$  with compact support in  $I$ , we have

$$\int_I |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |\xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (25)$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= |\alpha|^{p(x)-2} x^{\alpha(p(x)-\beta-1)-p(x)} \cdot \bar{g}(x) dx, \\ \mu_{2,\beta}(dx) &= x^{\alpha(p(x)-\beta-1)} \left( 2 \cdot \frac{p(x)-1}{\beta-\sigma(x)} \right)^{p(x)-1} dx. \end{aligned}$$

*Proof.* We apply Theorem 5.1 with the function  $u = x^\alpha$ . We note that  $u' = \alpha x^{\alpha-1}$  and  $u'' = \alpha(\alpha-1)x^{\alpha-2}$  and thus according to (22) we have

$$g(x) = x^{2\alpha-2} \left[ \sigma(x)\alpha^2 - p'(x)x\alpha \log |\alpha x^{\alpha-1}| + (p(x)-1)\alpha(1-\alpha) \right],$$

which is nonnegative due to (24). Direct computations gives inequality (25) with the desired measures.  $\square$

As an another example we give the following inequality, where we allow  $I = (0, \infty)$ .

**Corollary 5.3.** *Suppose  $I \subseteq \mathbb{R}_+$  is an open subset,  $p \in \mathcal{P}(I)$ , and  $a > 0$  is an arbitrary number. Assume further that  $\sigma(x)$  and  $\beta > 0$  satisfy condition (7) and the following condition is satisfied*

$$\bar{g}(x) := \sigma(x) + p'(x)x \log \frac{a}{x^2} - 2p(x) + 2 \geq 0 \quad \text{a.e. in } I.$$

Then for every Lipschitz function  $\xi$  with compact support in  $I$ , we have

$$\int_I |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |\xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (26)$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= \left( \frac{a}{x} \right)^{p(x)-\beta-1} x^{-p(x)} \cdot \bar{g}(x) dx, \\ \mu_{2,\beta}(dx) &= \left( \frac{a}{x} \right)^{p(x)-\beta-1} \left( 2 \cdot \frac{p(x)-1}{\beta-\sigma(x)} \right)^{p(x)-1} dx. \end{aligned}$$

*Proof.* We apply Theorem 5.1 with the function  $u = \frac{a}{x}$ . Direct computations gives inequality (26) with the desired measures.  $\square$

We obtain also an inequality with exponential-type weights, where we allow  $I = (0, \infty)$  as well as  $I = (-\infty, \infty)$ .

**Corollary 5.4.** *Suppose  $I \subseteq \mathbb{R}$  is an open subset,  $p \in \mathcal{P}(I)$ , and  $a > 0$  is an arbitrary number. Assume further that  $\sigma(x)$  and  $\beta > 0$  satisfy condition (7) and the following condition is satisfied*

$$\bar{g}(x) := \sigma(x) - p'(x)x - p(x) + 1 \quad \text{a.e. in } I.$$

*Then for every Lipschitz function  $\xi$  with compact support in  $I$ , we have*

$$\int_I |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |\xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I |\xi \log \xi|^{p(x)} \frac{|p'(x)|^{p(x)}}{p(x)^{p(x)}} \mu_{2,\beta}(dx), \quad (27)$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= \bar{g}(x) e^{x(p(x)-\beta-1)} dx, \\ \mu_{2,\beta}(dx) &= \left( 2 \cdot \frac{p(x)-1}{\beta-\sigma(x)} \right)^{p(x)-1} e^{x(p(x)-\beta-1)} dx. \end{aligned}$$

*Proof.* We apply Theorem 5.1 with the function  $u = e^x$ . We note that according to (22) we have

$$g(x) = e^{2x} (\sigma(x) - p'(x)x - p(x) + 1),$$

which is nonnegative by the assumption. Direct computations gives inequality (27) with the desired measures.  $\square$

**Remark 5.1.** *We give examples of triplets of  $p(x)$ , the interval  $I$ , and  $\sigma(x)$  admissible in Corollary 5.4.*

- *For arbitrary  $d > 0$ , we may take  $p(x) = 1 + \frac{d}{|x|+1}$ , any interval  $I \subseteq \mathbb{R}$ , and any function  $\sigma(x)$ , which is nonnegative and continuous on  $\bar{I}$ .*
- *We may take  $p(x) = e^x$ , any finite interval  $I \subseteq \mathbb{R}_+$ , and any function  $\sigma(x)$  continuous on  $\bar{I}$  such that*

$$\sigma(x) \geq (x+1)e^x - 1.$$

- *We may take  $p(x) = 2 - e^{-x^2}$ ,  $I = (0, \infty)$ , and  $\sigma(x) \geq e^{-x^2}(2x^2 - 1) + 1$  (e.g.  $\sigma(x) \equiv 2e^{-3/2} + 1$ ).*

## 6 Links with the existing results

In this section we present several applications of Theorem 4.1. We start with re-obtaining the main result of Skrzypczak [49], which deals with constant function  $p$  and implies classical Hardy inequality with optimal constant (see [49], Theorem 5.1). Then we concentrate on the comparison with the results of Harjulehto–Hästö–Koskenoja [20] and Mashiyev–Çekiş–Mamedov–Ogras [37]. We mention also the related papers considering inequalities involving Hardy operator.

In our paper [16] we focus on  $n$ -dimensional inequalities, in particular with radial weights.

### Results of Skrzypczak [49, 50]

When we consider  $1 < p(x) \equiv p < \infty$  in Theorem 4.1, we retrieve the main result of [49], implying the classical Hardy inequality with the optimal constant (see [49] for the details and the numerous other examples). Moreover, the following theorem leads to Hardy–Poincaré inequalities with the weights of a type  $\left(1 + |x|^{\frac{p}{p-1}}\right)^\alpha$ , where the constants are proven to be optimal for sufficiently big parameter  $\alpha > 0$  (see [50] for the details).

**Corollary 6.1** ([49, Theorem 4.1]). *Assume that  $1 < p < \infty$  and  $u \in W_{loc}^{1,p}(\Omega)$  is a nonnegative solution to the PDI  $-\Delta_p u \geq \Phi$ , in the sense of Definition 2.1, where function  $\Phi$  is locally integrable and satisfies the condition*

$$(\Phi, \mathbf{p}) \quad \sigma_0 := \inf \{ \sigma \in \mathbb{R} : \Phi \cdot u + \sigma |\nabla u|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\} \} \in \mathbb{R}. \quad (28)$$

*Assume further that  $\beta$  and  $\sigma$  are arbitrary numbers such that  $\beta > 0$  and  $\beta > \sigma \geq \sigma_0$ . Then, for every Lipschitz function  $\xi$  with compact support in  $\Omega$ , we have*

$$\int_{\Omega} |\xi|^p \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_{2,\beta}(dx),$$

where

$$\begin{aligned} \mu_{1,\beta}(dx) &= \left( \frac{\beta - \sigma}{p - 1} \right)^{p-1} (\Phi \cdot u + \sigma |\nabla u|^p) \cdot u^{-\beta-1} \chi_{\{u>0\}} dx, \\ \mu_{2,\beta}(dx) &= u^{p-\beta-1} \chi_{\{|\nabla u| \neq 0\}} dx. \end{aligned}$$

**Remark 6.1.** The paper [13] applies the results of [49] in order to obtain Poincaré inequalities with the best constants [13, Remark 7.6]. It is also proven therein that Hardy inequalities obtained in [49] lead to the solvability of certain family of degenerated PDEs like  $\operatorname{div}(\rho(x)|\nabla u(x)|^{p-2}\nabla u(x)) = x^*$ , where  $x^*$  is a functional on the weighted Sobolev space  $W_\rho^{1,p}(\Omega)$ , involving degenerated  $p$ -Laplacian [13, Theorem 7.12]. Moreover, the results of [49] enable to formulate alternative interpretation of the first eigenvalue of  $p$ -Laplacian [13, Remark 7.7].

### Results of Harjulehto–Hästö–Koskenoja [20]

Paper [20] concerns the  $n$ -dimensional norm version of Hardy-type inequality, but also the one-dimensional case is specially emphasized therein. Let us mention the following result.

**Theorem 6.1** ([20, Theorem 5.2]). *Let  $I = [0, M)$  for  $M < \infty$ , the variable exponent  $p : I \rightarrow [1, \infty)$  be bounded,  $p(0) > 1$  and*

$$\limsup_{x \rightarrow 0^+} (p(x) - p(0)) \log \frac{1}{x} < \infty.$$

*Moreover, suppose  $\operatorname{ess\,inf}_{x \in (0, x_0)} p(x) = p(0)$  for some  $x_0 \in (0, 1)$ .*

*If  $a \in [0, 1 - \frac{1}{p(0)})$ , then Hardy-type inequality*

$$\|\xi(x)x^{a-1}\|_{L^{p(x)}(I)} \leq C \|\xi'(x)x^a\|_{L^{p(x)}(I)} \quad (29)$$

*holds for every  $\xi \in W^{1,p(x)}(I)$  with  $\xi(0) = 0$ .*

We have the following related result.

**Corollary 6.2.** *Suppose  $I \subseteq \mathbb{R}_+$  is an open subset and  $a, \beta > 0$  are arbitrary numbers. Let  $p \in \mathcal{P}(I)$  and assume that there exists a continuous function  $A(x)$  such that*

$$a\beta + (1-a)xp'(x) \log x + (a-3)(p(x)-1) \geq A(x) \geq 0. \quad (30)$$

*Then, for every Lipschitz function  $\xi$  with compact support in  $I$ , we have*

$$\int_I |x^{a-1}\xi|^{p(x)} \mu_{1,\beta}(dx) \leq \int_I |x^a \xi'|^{p(x)} \mu_{2,\beta}(dx) + \int_I \left( x^a |\xi \log \xi| \frac{|p'(x)|}{p(x)} \right)^{p(x)} \mu_{2,\beta}(dx), \quad (31)$$

where

$$\begin{aligned}\mu_{1,\beta}(dx) &= x^{-a(\beta+1)} A(x) dx, \\ \mu_{2,\beta}(dx) &= x^{-a(\beta+1)} dx.\end{aligned}$$

*Proof.* We apply  $u = \frac{1}{a}x^a$  in Theorem 4.1 and we obtain inequality (16) with measures  $\tilde{\mu}_1, \tilde{\mu}_2$ . We simplify the right-hand side measure  $\tilde{\mu}_2$  by taking  $\sigma(x) = \beta - \frac{2}{a}(p(x) - 1)$  which satisfies crucial conditions.

We ensure the condition (6) by (30). Indeed, we estimate the expression in the left-hand side measure  $\tilde{\mu}_1$  from below as follows

$$\begin{aligned}& a\beta - 2(p(x) - 1) + (1 - a)[x \log xp'(x) - (p(x) - 1)] = \\ &= a\beta + (1 - a)x \log xp'(x) + (a - 3)(p(x) - 1) \geq A(x) \geq 0.\end{aligned}$$

We reach the goal by dividing both sides by  $a^\beta$ .  $\square$

**Remark 6.2** (Comparison of Theorem 6.1 and Corollary 6.2). Inequalities (29) and our (31) are similar, however there are some differences. Inequality (31) is a modular version, while (29) is a norm one and it involves the additional term as well as the weights  $\mu_{1,\beta}$  and  $\mu_{2,\beta}$  of power type with strictly negative exponents. The requirements on  $p(x)$  are of different types. Furthermore, we formulate inequality (31) for every Lipschitz and compactly supported function  $\xi$  in  $I$ , while (29) is stated for  $\xi \in W^{1,p(x)}(I)$  with  $\xi(0) = 0$ . Moreover, we allow infinite interval  $I$  and a bit different range of parameter  $a$ .

**Remark 6.3.** The following functions  $p(x)$  are admissible both in Theorem 6.1 and in Corollary 6.2. We note that in Theorem 6.1 we need to restrict our consideration to the interval  $I = (0, M)$ ,  $M < \infty$ . To compare with Corollary 6.2, in the two last examples we allow the infinite interval  $I$ .

- If  $\gamma > 1$ , we take  $p(x) = x + \gamma$ .
- If  $\gamma \geq 1$ , we take  $p(x) = 2 - \frac{1}{x+\gamma}$ .
- If  $\gamma > 0$  and  $d_1 > d_2 > 0$ , we take  $p(x) = 1 + \frac{\gamma+d_1x}{\gamma+d_2x}$ .

In every example of the above ones, in our (31) in  $\mu_{1,\beta}$  we may choose  $A(x)$  separated from zero.

The result of [20] was further developed in variable exponent Orlicz–Sobolev setting [32].

### Results of Mashiyev–Çekiç–Mamedov–Ogras [37]

In [37] the authors prove the following extension of Hardy–type inequality from [20] by Harjulehto–Hästö–Koskenoja described above.

**Theorem 6.2** ([37, Theorem 3]). *Suppose  $p(x)$ ,  $q(x)$  and  $\alpha(x)$  are log–Hölder continuous at the origin and at the infinity, i.e. there exist constants  $C_i$ ,  $i = 1, 2$ , such that the following conditions hold*

$$|p(x) - p(0)| \log \frac{1}{x} \leq C_1, \quad \text{where } x \in (0, 1/2]$$

and

$$|p(x) - \lim_{|x| \rightarrow \infty} p(x)| \log(e + x) \leq C_2, \quad \text{where } x \in (0, \infty),$$

with  $1 < p^- \leq p(x) \leq q(x) \leq q^+ < \infty$  and  $-\infty < \alpha^- \leq \alpha(x) < \infty$  for  $x \in (0, \infty)$ . Then there exists a constant  $C > 0$  such that for every function  $\xi$ , absolutely continuous on  $[0, \infty)$ , with  $\xi(0) = 0$  we have

$$\|\xi(x)x^{\alpha(x) - \frac{1}{p'(x)} - \frac{1}{q(x)}}\|_{L^{q(x)}(0, \infty)} \leq C \|\xi'(x)x^{\alpha(x)}\|_{L^{p(x)}(0, \infty)}. \quad (32)$$

In [20] the authors prove (32) with constant  $\alpha$ ,  $q(x) = p(x) > 1$ , on a finite interval  $I$ , and without the assumption  $p(0) \leq p(x)$  for small  $x$ 's.

We have the following related result.

**Remark 6.4.** When in Corollary 6.2 we assume additionally that  $A(x)$  has values separated from zero, we take  $I = \mathbb{R}_+$ , we put  $\alpha(x) = a \left(1 - \frac{\beta+1}{p(x)}\right)$ , and we rearrange power–type terms, we obtain

$$A_0 \left( \int_0^\infty |x^{\alpha(x)-1} \xi|^{p(x)} dx \right) \leq \int_0^\infty |x^{\alpha(x)} \xi'|^{p(x)} + \left( x^{\alpha(x)} |\xi \log \xi| \frac{|p'(x)|}{p(x)} \right)^{p(x)} dx$$

for every Lipschitz function  $\xi$  with compact support in  $\mathbb{R}_+$ .

The comparison of Theorem 6.2 with our above inequality is similar as in the case of theorem by Harjulehto–Hästö–Koskenoja [20] (see Remark 6.2).

### Results of Diening–Samko [15], Rafeiro–Samko [44], Harman [22] and others

In [15] the derived Hardy–type inequality involves Hardy operator. For  $p \in \mathcal{P}(0, \infty)$  satisfying conditions related to log–Hölder continuity, the authors

prove the following inequality

$$\left\| x^{\alpha(x)+\mu(x)-1} \int_0^x \frac{\xi(y)}{y^{\alpha(y)}} dy \right\|_{L^{q(x)}(0,\infty)} \leq C \|\xi\|_{L^{p(x)}(0,\infty)},$$

where  $x \in (0, \infty)$ , an exponent  $q \in \mathcal{P}(0, \infty)$  is any function such that  $\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0)$  with  $\mu(0) \in [0, \frac{1}{p(0)})$ ,  $\frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty)$  with  $\mu(\infty) \in [0, \frac{1}{p(\infty)})$  and  $\alpha(0) < \frac{1}{p'(0)}$ ,  $\alpha(\infty) < \frac{1}{p'(\infty)}$ . Exponents  $q(x)$ ,  $\mu(x)$  and  $\alpha(x)$  are also supposed to satisfy local log-Hölder condition in zero and infinity.

In [44] by Rafeiro-Samko the derived Hardy-type inequality involves the Riesz potential. It is stated on a bounded domain  $\Omega \subset \mathbb{R}^n$ , which complement has the cone property. Similar Hardy-type inequality is considered in [33].

An inequality corresponding to results of [44], but involving Hardy operator  $Hv(x) = \int_0^x v(t)dt$ , is proven in [22]. The authors derive the following inequality which holds for every nonnegative and locally integrable function  $\xi$

$$\|H\xi|x|^{\alpha(x)-1}\|_{L^{p(x)}(0,l)} \leq \|\xi|x|^{\alpha(x)}\|_{L^{p(x)}(0,l)}, \quad (33)$$

where  $l > 0$ , functions  $\alpha, p : (0, l) \rightarrow \mathbb{R}$  are measurable and such that  $-\infty < \alpha^- \leq \alpha(x) \leq \alpha^+ < \infty$  and  $-\infty < p^- \leq p(x) \leq p^+ < \infty$ . Moreover, the author indicate the necessary condition for validity of Hardy inequality (33) (see [22, Theorem 3, 4]).

There are several other papers dealing with one-dimensional Hardy inequality involving Hardy operator, e.g. [11, 23, 24, 25, 34, 35]. Those papers consider the further regularity analysis of inequality similar to (33), with different kind of weights under norm. The authors indicate the different type of regularity in the neighborhood of zero and at infinity for the variable exponents and present the necessary and sufficient conditions for the validity of Hardy inequality.

## 7 Open questions

We find it interesting to investigate the following ideas.

### Erasing the additional term

Is it possible to improve (16) to an inequality of the following form

$$\int_{\Omega} |\xi|^{p(x)} \mu_{1,\beta}(dx) \leq c_2 \int_{\Omega} |\nabla \xi|^{p(x)} \mu(dx),$$

where  $c_2 > 0$  and  $\mu_{1,\beta}(dx)$  is given by (17), and  $\mu(dx)$  is eventually worse than  $\mu_{2,\beta}(dx)$  given by (18)?

### Improving the right-hand exponent

We find it deserving attention to improve an exponent on the right-hand side of (16). When is it possible to prove an inequality

$$\int_{\Omega} |\xi|^{q(x)} \mu_{1,\beta}(dx) \leq \int_{\Omega} |\nabla \xi|^{p(x)} \mu_{2,\beta}(dx) + \int_{\Omega} |\xi \log \xi|^{p(x)} \mu_{3,\beta}(dx),$$

with  $q(x) > p(x)$ ?

### Inequalities in more general spaces

Investigation on Hardy inequalities in variable exponent spaces is lively studied topic. They are considered for Orlicz–Sobolev functions with  $|\nabla u| \in L^{p(\cdot)} \log L^{p(\cdot)q(\cdot)}$  in the literature [32]. Our framework is already applied in Orlicz setting [51] with constant type of growth. What inequalities can be obtained in variable exponent Orlicz–Sobolev via our method?

### Acknowledgments

The authors would like to thank Agnieszka Kałamajska, Tomasz Adamowicz, and Lech Maligranda for discussions and help in finding appropriate literature.

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